Nonlinear Fourier Transform at Defocusing Regime

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Continuous-time Model

In fiber-optic communication pulse propagation is well-modeled by the *stochastic nonlinear Schrödinger (NLS) equation*

$$\frac{\partial q(t,z)}{\partial z} = \underbrace{-\frac{j\beta_2}{2}\frac{\partial^2 q(t,z)}{\partial t^2}}_{\text{dispersion}} + \underbrace{js\gamma|q(t,z)|^2 q(t,z)}_{\text{nonlinearity}} + \underbrace{n(t,z)}_{\text{AWGN}}. \quad (1)$$

- z is the distance along the fiber and t is time
- $q(t,z)$ is the signal complex envelope
- β_2 is the *chromatic dispersion* coefficient
- γ is the *nonlinearity parameter*
- $n(t,z)$ is the *white Gaussian noise*
- s could be 1 or -1, representing the *defocusing* (dark soliton) and *focusing* (bright soliton) regimie respectively.

Throughout propagation over an optical fiber, stochastic effects (noise), linear effects (dispersion) and nonlinear effects (Kerr nonlinearity) **interact**.
Even in the absence of noise, solving the NLS equation requires **numerical techniques** for partial differential equations (PDEs).

Nonlinear Fourier Transform (NFT)

Nonlinear Fourier Transform (NFT) of a signal $q(t)$ is defined via the spectral analysis of the L operator, given by [1]

$$L = j\begin{pmatrix} \frac{\partial}{\partial t} & -q(t,z) \\ sq^*(t,z) & -\frac{\partial}{\partial t} \end{pmatrix}$$

The spectrum of L is found by solving the eigenproblem

$$Lv = \lambda v,$$

where λ is an eigenvalue of L and v is its associated eigenvector. It can be shown that the operator L has the *isospectral flow* property, *i.e.*, its spectrum is invariant evn as q evolves according to the NLS equation.
The eigenproblem $Lv = \lambda v$ can be simplified to Zakharov-Shabat system

$$v_t = P(\lambda, q)v = \begin{pmatrix} -j\lambda & q(t) \\ sq^*(t) & j\lambda \end{pmatrix} v$$

The NFT of $q(t)$ consists of continuous and discrete spectral functions $\hat{q}(\lambda)$ and $\tilde{q}(\lambda)$ where

$$\hat{q}(\lambda) = \frac{b(\lambda)}{a(\lambda)}, \quad \tilde{q}(\lambda) = \frac{b(\lambda_j)}{a_\lambda(\lambda_j)}, \quad j = 1,2,3,\dots,N.$$

in which λ_j are the zeros of $a(\lambda)$. Here $a(\lambda)$ and $b(\lambda)$ are given by

$$a(\lambda) = \lim_{t\to\infty} v_1 e^{j\lambda t},$$
$$b(\lambda) = \lim_{t\to\infty} v_2 e^{j\lambda t},$$

where v is a solution of Zakharov-Shabat system under the boundary condition

$$v(t,\lambda) \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-j\lambda t}, \quad t \to -\infty.$$

Only defocusing regime is considered in our work.
Two different numerical methods were used to compute NFT and INFT, *i.e.*, *modified Albowitz-Ladik* (modified AL method)and *Layer-Peeling* method (LP method).

Obtaining $a(\lambda)$ and $b(\lambda)$ from Continuous Spectrum

- $a(\lambda)$ is analytic in $\mathbb{C}^+$, and $|a(\lambda)|$ vanishes faster than $1/|z|$ as $z \to \infty$, therefore $\angle a(\lambda)$ and $log(|a(\lambda)|)$ is the *HilbertTransform* of each other.
- In this poster, we consider the **defocusing regime**

$$|a(\lambda)| = \left(\frac{1}{1 - |\hat{q}(\lambda)|^2}\right)^{\frac{1}{2}},$$

$$\angle a(\lambda) = \mathcal{H}(log(|a(\lambda)|)).$$

NFT and INFT with Modified AL Method

The **forward AL iteration** equation is

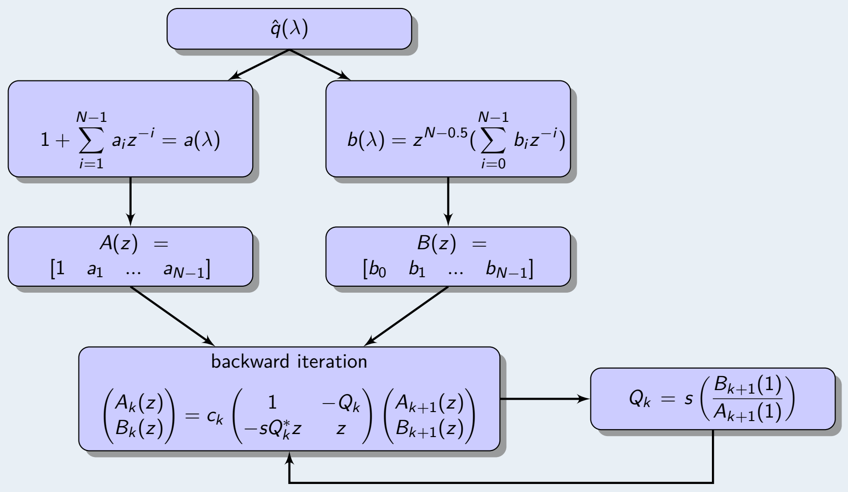
$$v[k+1] = c_k \begin{pmatrix} z & Q[k] \\ sQ[k]^* & z^{-1} \end{pmatrix} v[k]$$

where $z = e^{-j\lambda\epsilon}$, $Q[k] = q[k]\,\epsilon$ and $c_k = \frac{1}{\sqrt{1 - s|Q[k]|^2}}$.

- for numerical stability, we require the *applicability condition*

$$\text{where } L = \left|\prod_{k=1}^{N}(1 - s|Q[k]|^2)\right|^{-\frac{1}{2}},$$
then we require that
$$k_1 \leqslant L \leqslant k_2$$

so that L is not too large or too small. Ideally L should be near one. We define $(A_k, B_k) = (a_k, z^{-2k+1}b_k)$, $z = e^{-2j\lambda\epsilon}$ and substitute $z^2 \to z$, the **inverse AL iteration** procedure [2] is as follows



- **sensitivity issues** when $Q[k] \approx 1$, 1% change of $v[k]$ will lead to dramatic changes of $v[k-1]$.
- $(1 - s|Q[k]|^2) = 0$ will lead to ill-conditioned matrix.
- Details are refered to [2].

NFT and INFT with LP Method

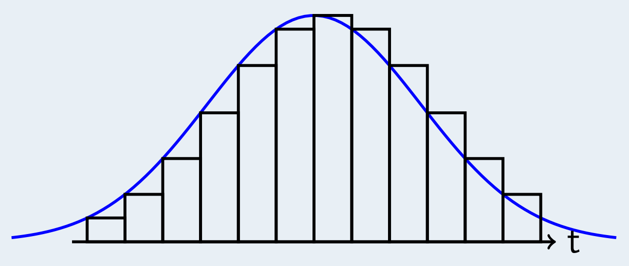
As showed in the figure below, at each iteration we combine the NFT of rectangular pulse with the NFT of signal from $t = -\infty$ to that moment.
In INFT, the **backward LP iteration** is

$$u_k = M_k^{-1} u_{k+1} = \begin{pmatrix} \bar{x}_k(\lambda) & -\bar{y}_k(\lambda) \\ -y_k(\lambda) & x_k(\lambda) \end{pmatrix} u_{k+1}, \quad u_N = \begin{pmatrix} a(\lambda) \\ b(\lambda) \end{pmatrix}$$

$$x_k(\lambda) = (cos(\triangle\epsilon) - j\frac{\lambda}{\triangle}sin(\triangle\epsilon))e^{j\lambda(t_k - t_{k-1})},$$
$$\bar{y}_k(\lambda) = \frac{q_k}{\triangle}sin(\triangle\epsilon)e^{j\lambda(t_k + t_{k+1})},$$
$$y_k(\lambda) = \frac{sq_k^*}{\triangle}sin(\triangle\epsilon)e^{-j\lambda(t_k + t_{k+1})},$$
$$\bar{x}_k(\lambda) = (cos(\triangle\epsilon) + j\frac{\lambda}{\triangle}sin(\triangle\epsilon))e^{-j\lambda(t_k - t_{k-1})}.$$

where $\triangle = \sqrt{\lambda^2 - s|q_k|^2}$, and in the inverse propagation q_k is approximated by q_{k+1}.
Since q(T)=0, from ([1],Part I, Eq. 32) and $V^1(T,\lambda) = [0 \quad 1]^T$:

$$q^*(T^-) = s\frac{1}{\pi}\int_{-\infty}^{\infty}\hat{q}(\lambda)e^{2j\lambda T^-}d\lambda$$



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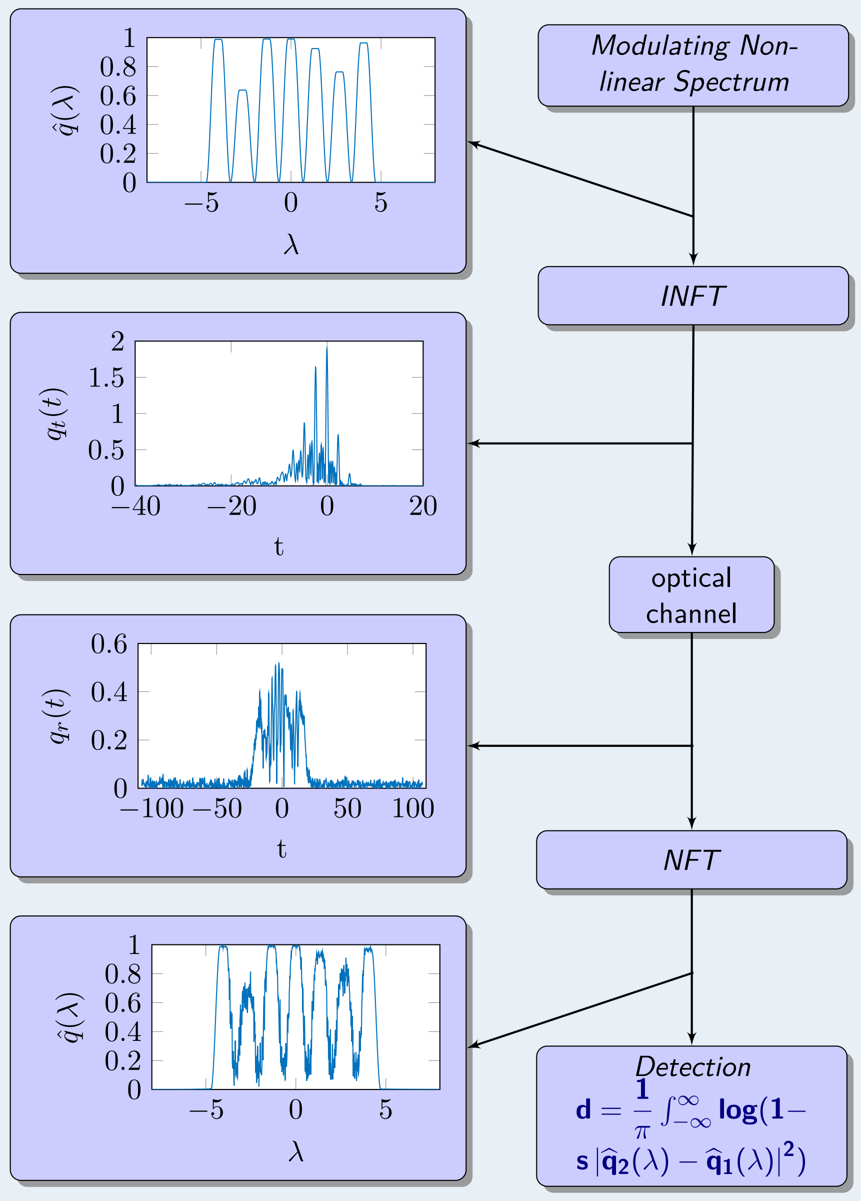
Layer-peeling method

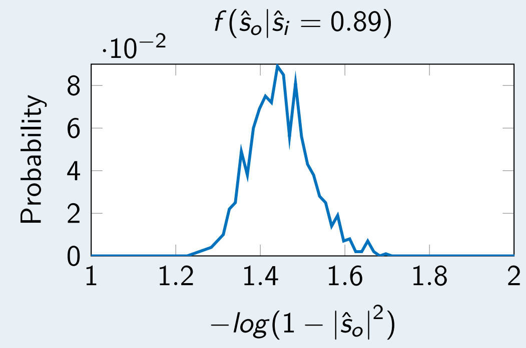
Channel Capacity Estimation

- A simulation was designed to estimate the channel capacity of optical fiber.
- The total nonlinear bandwidth was devided into 7 users. Each user has a raised-cosine pulse with amplitude ranging from 0.5 to 0.99 into 32 levels geometrically. It gives signal at each user 5 bits.
- The maximal noise bandwidth occurs when 7 users have the maximal energy.
- A fine quantization was used at the output, giving a smooth probability solution.
- *Log-euclidean metric* is used on detection.

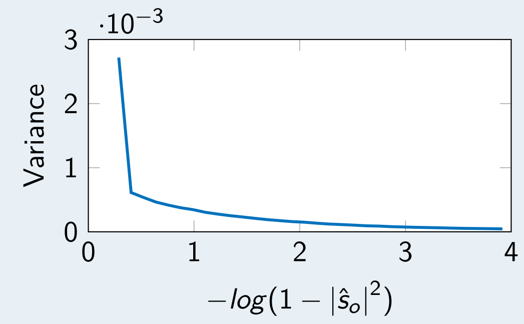
Channel Capacity Estimation (continued)



- The central user has amplitude of 0.89 and the others are chose uniformly randomly.



- Variances of the outputs after detection are compared.



- The data rate is **3.43 bit/symbol**.
- Spectrum efficiency and SNR is yet to be calculated.

Future Work

- *Nonlinear Frequency-devision Multipluxing* (NFDM) in the focusing regime.
- Capacity of NFDM.
- Higher spectrum efficiency and data rate.
- Stability of numerical method.

References

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